Green's Function Methods and the Nonhomogeneous Wave Equation

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Preliminaries

- Motivating Scenario: Suppose we want to take the acoustic characteristics (or reverb) of a room, isolate these characteristics, and store them for later application to any number of audio signals
- Goal: Examine the mathematical machinery of Green's Functions, which is required to produce some data set which allows us to virtually "place" any given audio signal in the room in question, ultimately producing a final result which sounds like the audio signal as it would have sounded had it been recorded in the original room
- Additionally, we consider the limitatons imposed by the spatially dependent nature of these Green's functions and seek to identify domain geometries where this local dependence may be overcome by interpolation

Methodology

We first establish the following:

Definition

Consider some linear time-invariant differential operator L and some domain Ω, and define the nonhomogeneous differential equation on Ω by the following:

$$Lu(x) = f(x)$$

with some given boundary conditions. A **Green's Function for L on Ω** is a function $G(x, y)$ (or more generally, a distribution) satisfying:

$$LG(x, y) = \delta_0(x - y)$$

Further, G must satisfy the given boundary conditions $\vec{\varphi}$ of the original differential equation.

If $\vec{\varphi} = \vec{0}$ (homogeneous Diriclet boundary conditions), then, exploiting linearity of the operator L, we obtain

$$u(x) = \int_\Omega G(x, y) f(y)\, dy$$

Clearly, if G is known for operator L on domain Ω, then we may obtain u for any reasonably defined f. Therefore, the heart of the methodology of Green's functions centers on obtaining G for a given L and domain Ω. For our purposes, we will simply seek solutions to $Lu(x) = f(x)$ of the form $u(x) = \int_\Omega G(x, y) f(y)\, dy$ and we allow the domain Ω to depend on time t.

A Sample Problem

As a sample problem, we consider

$$(\dagger)\begin{cases} \partial_t^2 u - \partial_x^2 u = F(x,t) & \forall (x,t) \in [0,1] \times [0,\infty) \\ u(x,0) = 0 = \partial_t u(x,0) & \forall x \in [0,1] \\ u(0,t) = 0 = u(1,t) & \forall t \in [0,\infty) \end{cases}$$

An application of Duhamel's Principle yields the corresponding homogeneous Initial-Boundary Value Problem:

$$(\ddagger)\begin{cases} \partial_t^2 V - \partial_x^2 V = 0 & \forall (x,t) \in [0,1] \times [0,\infty) \\ V(x,0) = 0 \quad \text{and} \quad \partial_t V(x,0) = f(x) & \forall x \in [0,1] \\ V(0,t) = 0 = V(1,t) & \forall t \in [0,\infty) \end{cases}$$

If we consider

$$V(x,t) = T\left[f(\xi)\right](x,t)$$

where T is the linear operator which transforms $f = \partial_t u$ into u. Then the solution to our original problem $(\dagger)$ is given by

$$u(x,t) = \int_0^t T\left[F(\xi,\tau)\right](x, t-\tau)\, d\tau$$

We shall assume that f and F are sufficiently well-behaved, say $f, F \in L^2$, so that their Fourier sine series expansions on $[0,1]$ in x converge in some sense, and further that f, F are supported on $S \subseteq [0,1]$.

Results

We solve $(\ddagger)$ using separation of variables. Exploiting the property that $f(x) = \sum_{n=1}^\infty b_n \sin(n\pi x)$ for all $x \in [0,1]$, where $b_n = \frac{\langle f, \sin(n\pi \cdot)\rangle}{\langle \sin(n\pi\cdot), \sin(n\pi\cdot)\rangle}$, this yields

$$V(x,t) = T\left[f\right](x,t) = \sum_{n=1}^\infty \left(\frac{\sin(n\pi t)}{n\pi}\right) b_n \sin(n\pi x)$$

Further, since f, F are compactly supported on a subset of $[0,1]$, we may consider the odd extensions of f, F to $[-1,1]$ in x, denoted f^o, F^o to deduce that for $x, t \in [0,1]$

$$V(x,t) = T\left[f^o\right](x,t) = \sum_{n=1}^\infty \left(\frac{\sin(n\pi t)}{n\pi}\right) b_n \sin(n\pi x) \sim \left(f^o * g(\cdot, t)\right)(x)$$

with $g(x,t) = \Theta(x+t) - \Theta(x-t)$, where Θ denotes the Heaviside Function. (Note $\hat{g}(n,t) = \frac{\sin(n\pi t)}{n\pi}$ for $n \in \mathbb{Z} \setminus \{0\}$.) Thus, applying Duhamel's Principle and exploiting the properties of integrals of odd functions on intervals centered at 0, we obtain

$$\begin{aligned} u(x,t) &= \int_0^t T\left[F^o\right](x, t-\tau)\, d\tau = \int_0^t \left(F^o * g(\cdot,t)\right)(x, t-\tau)\, d\tau \\ &= \int_0^t \int_{-1}^1 F^o(y, t-\tau)\left[\Theta(x - y + (t-\tau)) - \Theta(x - y - (t-\tau))\right] dy\, d\tau \\ &= \iint_{\Omega_t = [0,1]\times[0,t]} F(y,\sigma)\left[\frac{\Theta(x+y-\sigma) + \Theta(x-y+\sigma) - \Theta(x+y+\sigma) - \Theta(x-y-\sigma)}{2}\right] dy\, d\sigma \end{aligned}$$

Thus, if we allow the domain Ω in our definition of Green's functions to depend on t, we have written u in the desired form to identify G as follows for $x, y \in (0,1)$ and $\sigma \in [0,t]$:

$$G(x,y;t,\sigma) = \frac{\Theta(x+y-\sigma) + \Theta(x-y+\sigma) - \Theta(x+y+\sigma) - \Theta(x-y-\sigma)}{2}$$

Some Implications

- The dependence of Ω_t on t raises the question: "Is it possible to find G in a manner which depends directly on t such that the integration domain Ω is independent of t?"
- The dependence of G on the choice of y in this bounded domain indicates that in application scenarios, measured impulse responses depend on the position of the source in the domain, leading naturally to questions of interpolating or extrapolating several position-dependent Green's functions to facilitate the exact or approximate calculation of Green's functions at other points in the domain
- If such interpolation or extrapolation schemes are possible, it is then natual to seek the lower bound on the number of measurements required for an accurate interpolation for a given class of domains

Avenues of Future Work

- Seek a closed form expression for G for the sample problem where the domain Ω is independent of t
- Examine the spacial extension of this methodology to subsets of $\mathbb{R}^n$, and in partucular to domains in $\mathbb{R}^n$ where the boundaries are piecewise defined by manifolds
- Examine connections of the geometric and spectral properties of the domain to determine if there are viable conditions for constructing an interpolation scheme for measured (and hence spatially dependent) impulse responses
- Seek generalizations of Green's functions to allow for more complications at the boundary of the domain (for example, different boundary condtions)
- Experiment with analogous techniques in nonlinear PDE cases

References

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- Walter A. Strauss, *Partial Differential Equations: An Introduction.* John Wiley & Sons, 1992.
- H. F. Weinberger, *A First Course in Partial Differential Equations.* Blaisdell Publishing Company, 1965.