Nonstandard Methods Applied to the AIM Conjecture

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Mathematics

Goals of Research

- **Goal**—using nonstandard methods, prove if $\mathbb{L}$ is a loop with arbitrarily large finite AIM loops, then $\mathbb{L}$ is AIM
- **Context**—proof of AIM Conjecture has been attempted using multiple methods for more than a decade

Loop & Subloop Def.

- **Loop** $\mathbb{L}$—algebraic structure with:
  - underlying set L of $\mathbb{L}$, identity element 1 & binary operation $\cdot^{\mathbb{L}}$
  - $\forall\, a, b \in L$, $a \cdot^{\mathbb{L}} x = c$ & $y \cdot^{\mathbb{L}} b = d$ have solutions in L
- **Equivalent definition**—algebra with binary operators $\cdot^{\mathbb{L}}$, $/^{\mathbb{L}}$ & $\backslash^{\mathbb{L}}$ where:
  - $x \backslash^{\mathbb{L}} (x \cdot^{\mathbb{L}} y) = y$
  - $x \cdot^{\mathbb{L}} (x \backslash^{\mathbb{L}} y) = y$
  - $(x \cdot^{\mathbb{L}} y) /^{\mathbb{L}} y = x$
  - $(x /^{\mathbb{L}} y) \cdot^{\mathbb{L}} y = x$
  - $1^{\mathbb{L}} \cdot^{\mathbb{L}} x = x \cdot^{\mathbb{L}} 1^{\mathbb{L}} = x$
- **Subloop** $\mathbb{A}$ **of** $\mathbb{L}$—$A \subseteq L$ & operations on A are operations on L restricted to A

Translations Def.

- **Left & right translations of loop** $\mathbb{L}$—self maps of the forms L_x & R_x resp. for $x \in L$ where $L_x(y) = xy$ & $R_x(y) = yx$
- **Multiplication group of loop** $\mathbb{L}$—group with underlying set of left & right translations on $\mathbb{L}$

AIM Conjecture

- **Nucleus of loop** $\mathbb{L}$ **Nuc**($\mathbb{L}$)—set of all elements of L that associate with all other elements of L
- **Center of loop** $\mathbb{L}$—set of all elements in $\mathrm{Nuc}(\mathbb{L})$ that commute with all elements of L
- **AIM**—abelian inner-mapping
- **AIM Conjecture**—posits when loop $\mathbb{L}$ has abelian $\mathrm{Inn}(\mathbb{L})$, then:
  - $\mathbb{L}/\mathrm{Nuc}(\mathbb{L})$ is an abelian group
  - $\mathbb{L}/\mathrm{Z}(\mathbb{L})$ is a group

Equivalent Statement

- **AIM Conjecture** can be proven for loop $\mathbb{L}$ by showing the following:
  - $a(a(x,y,z),u,w) = 1$
  - $K(a(x,y,z),u) = 1$

  where
  - $K(x,y) = (yx)\backslash(xy)$
  - $a(x,y,z) = (x \cdot yz)\backslash(xy \cdot z)$
- Originally seven equations, but they are equivalent to the first two

- **Inner-mapping group of** $\mathbb{L}$**, Inn**($\mathbb{L}$)—subset of multiplication group that fixes identity element 1

Nonstandard Methods

- **Nonstandard methods**–use superstructure embeddings to transfer first-order sentences to:
  - extend results
  - apply these results onto the original structures
- **First-order sentences**—logical formulas with well-defined truth values using universal & existential quantifiers over entire sets
- **Example**—**Spillover Principle** lets us draw conclusions about infinite structures/elements given results involving arbitrarily large finite structures/elements
- **Superstructure embedding** *—given two sets of urelements U & $^*U$ with injective function $^*: V(U) \longrightarrow {^*V}({^*U})$ where $a \in b \iff {^*a} \in {^*b} \implies V(U)$ is an elementary substructure of $^*V({^*U})$) up to isomorphism
- **Superstructure of** $\boldsymbol{U}$—$V(U) = \bigcup_{i=0}^{\infty} V_i(U)$ for set U of **urelements** with $V_0(U) = U$ & $V_{i+1}(U) = V_i(U) \cup \wp(V_i(U))$
- **Transfer Principle**—elementary embedding extends to a syntactic transform on first-order formulas: $\varphi \iff {^*\varphi}$

Results

- **Proved** that if $\mathbb{L}$ is a loop with arbitrarily large finite AIM subloops, then $\mathbb{L}$ is an AIM loop
- **Used nonstandard methods** to limit the collection of loops under consideration

Conclusion

- **Key terms**—loop, AIM Conjecture & nonstandard methods
- **Result**—loop $\mathbb{L}$ with arbitrarily large finite AIM subloops $\implies \mathbb{L}$ is AIM
  - **Tools**—nonstandard methods
- **Future goals**:
  - Prove arbitrarily large finite AIM loops $\implies$ an AIM loop structure can be defined on any set of urelements
  - Prove arbitrarily large finite AIM loops $\implies$ every loop is an AIM loop

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